# Capacity of an Extension of Cover's Two-Look Gaussian Channel

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*Abstract* — **We extend Cover's two-look Gaussian channel to *dispersive*, linear, time-invariant channels. An arbitrary number of *colored*, additive, stationary, Gaussian noise/interference sources is considered. Each noise/interference source may cause correlated or uncorrelated components observed by the two looks. The novelty of this work is a capacity formula derived using the asymptotic eigenvalue distribution of block-Toeplitz matrices as well as the application of this result to wireline communications.**

## I. Extension of the two-look Gaussian channel

We extend the scalar two-look Gaussian channel ([1], p. 264) $\begin{bmatrix}Y_1\\Y_2\end{bmatrix}=X+\begin{bmatrix}N_1\\N_2\end{bmatrix}$, $X\sim\mathcal{N}\left(0,\sigma_x^2\right)$, $\begin{bmatrix}N_1\\N_2\end{bmatrix}\sim\mathcal{N}\left(\mathbf{0},\sigma_n^2\begin{bmatrix}1&\rho\\\rho&1\end{bmatrix}\right)$, whose capacity is $C=\frac{1}{2}\,\mathrm{ld}\left(1+\frac{2\sigma_x^2}{\sigma_n^2(1+\rho)}\right)$ bits per use, to the dispersive case of channel length $N$ with $K$ independent noise sources. Each of the two looks consists of $L$ random variables, representing $T$-spaced samples, that are stacked in $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$, respectively:

$$\underbrace{\begin{bmatrix}\boldsymbol{y}_1\\\boldsymbol{y}_2\end{bmatrix}}_{\boldsymbol{y}}=\underbrace{\begin{bmatrix}\boldsymbol{H}_1^{(x)}\\\boldsymbol{H}_2^{(x)}\end{bmatrix}}_{\boldsymbol{H}_x}\boldsymbol{x}+\underbrace{\begin{bmatrix}\boldsymbol{H}_1^{(n_1)}&\cdots&\boldsymbol{H}_1^{(n_K)}\\\boldsymbol{H}_2^{(n_1)}&\cdots&\boldsymbol{H}_2^{(n_K)}\end{bmatrix}}_{\boldsymbol{H}_n}\underbrace{\begin{bmatrix}\boldsymbol{n}_1\\\vdots\\\boldsymbol{n}_K\end{bmatrix}}_{\boldsymbol{n}}. \quad (1)$$

The matrices $\boldsymbol{H}_x$ and $\boldsymbol{H}_n$ consist of Toeplitz blocks. We consider continuous transmission, hence $L\to\infty$. In order to calculate the power-constrained channel capacity, we make the following assumptions: a) The receiver has perfect channel knowledge. b) The transmitter has no channel knowledge, hence the covariance matrix of the zero-mean Gaussian input vector $\boldsymbol{x}$ is $\boldsymbol{C}_{xx}=\sigma_x^2\boldsymbol{I}$. c) All elements of the Gaussian noise vector $\boldsymbol{n}$ are uncorrelated, have zero mean, and their covariance matrix is $\boldsymbol{C}_{nn}=\mathrm{diag}\left\{\left[\sigma_{n,1}^2\cdots\sigma_{n,K}^2\right]\right\}\otimes\boldsymbol{I}_{L+N-1}$. d) All random vectors and convolution matrices are assumed to be real-valued.

## II. Capacity Formula

The capacity in bits per second of the extended two-look Gaussian channel, defined in the previous section, is given by

$$\begin{aligned}C \;=\;& \tfrac{1}{2T}\Big(\tfrac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=1}^{2}\mathrm{ld}\left(\lambda_k\left(\boldsymbol{T}_{(1)}(\omega)\right)\right)\mathrm{d}\omega\\&-\tfrac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=1}^{2}\mathrm{ld}\left(\lambda_k\left(\boldsymbol{T}_{(2)}(\omega)\right)\right)\mathrm{d}\omega\Big),\end{aligned} \quad (2)$$

where $\lambda_k(\cdot)$ are the eigenvalues of

$$\boldsymbol{T}_{(i)}(\omega)\hat{=}\begin{bmatrix}t_{(i)}^{1,1}(\omega)&t_{(i)}^{1,2}(\omega)\\t_{(i)}^{2,1}(\omega)&t_{(i)}^{2,2}(\omega)\end{bmatrix},$$

and $t_{(i)}^{u,v}(\omega)\hat{=}\sum_{k=-\infty}^{\infty}t_{(i),k}^{u,v}e^{-j\omega k}$ are the Fourier transforms of the absolutely summable sequences $\{t_{(i)}^{u,v}\}=\{t_{(i),-(L-1)}^{u,v},t_{(i),-(L-2)}^{u,v},\cdots,t_{(i),L-1}^{u,v}\}$ defined by the two symmetric block matrices $\boldsymbol{H}_x\boldsymbol{C}_{xx}\boldsymbol{H}_x^{\mathrm{T}}+\boldsymbol{H}_n\boldsymbol{C}_{nn}\boldsymbol{H}_n^{\mathrm{T}}=\boldsymbol{\mathcal{T}}_L(\{t_{(1)}^{u,v}\})$ and $\boldsymbol{H}_n\boldsymbol{C}_{nn}\boldsymbol{H}_n^{\mathrm{T}}=\boldsymbol{\mathcal{T}}_L(\{t_{(2)}^{u,v}\})$. This result is obtained using the asymptotic eigenvalue distribution of block-Toeplitz matrices following *Theorem 3* of [2]. Results for the corresponding case with continuous random processes can be derived based on [3].

## III. Application to wireline communications

Our capacity result (2) can be applied to data transmission over the copper twisted pair in the following manner. At physical layer level, the signal is transmitted as a voltage difference between the two wires, i.e., as a differential-mode signal. Without changing the transmitted signal, we obtain an additional observation by measuring the common-mode signal at the receiver, which is defined as the arithmetic mean of two voltages between each wire and earth[2]. This scenario is modeled as a two-look channel, where $\boldsymbol{y}_1$ is the differential-mode signal and $\boldsymbol{y}_2$ is the common-mode signal.

Under realistic assumptions for the expected strong crosstalk in VDSL (very high speed digital subscriber line) systems [4], we found that the capacity of the two-look channel can exceed the capacity of the single-look channel roughly by a factor of two. The case with the greatest gain is when there is a dominant interferer and low noise power levels on both looks.

## References

[1] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, John Wiley and Sons, ISBN 0-471-06259-6, 1991.

[2] H. Gazzah, P. A. Regalia, and J.-P. Delmas, "Asymptotic Eigenvalue Distribution of Block-Toeplitz Matrices and Application to Blind SIMO Channel Identification," *IEEE Trans. Inform. Theory*, vol. 47, no. 3, pp. 1243–1251, Mar. 2001.

[3] M. S. Pinsker, *Information and Information Stability of Random Processes*, Holden Day, 1964.

[4] ETSI TM6, "Transmission and Multiplexing (TM); Access transmission systems on metallic access cables; Very high speed Digital Subscriber Line (VDSL); Part 1: Functional requirements," *TS 101 270-1, Version 1.1.6*, Aug. 1999.

[1] This work was done in part while Thomas Magesacher was with the Telecommunications Research Center Vienna (ftw.).

[2] Although one could in principle envisage to also transmit a common-mode signal, this would cause compatibility problems due to electromagnetic emissions of the twisted pair.